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Analysis and development of subgrid turbulence models preserving the symmetry properties of the Navier–Stokes equations

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Abstract

Navier–Stokes equations have fundamental properties such as the invariance under some transformations, called symmetries, which play an important role in the description of the physics of the equations (conservation laws, wall laws, ...). It is essential that turbulence models do not violate these properties. An analysis of a large set of subgrid models, according to the respect of the symmetries of Navier–Stokes equations is then done. As this analysis reveals that only few of the existing models are consistent with the symmetries, a new class of compatible models is proposed. This class is refined such that the models also respect the second law of thermodynamics. Finally, it is shown that this thermodynamic requirement leads to the stability of the model.

Keywords: Turbulence modeling; Symmetry group; Thermodynamics

1. Introduction

Currently, there exists a large number of turbulence models based on various mathematical or physical hypothesis (see [1]) for large-eddy simulation (LES). While many of these models give encouraging results, a major portion fail to meet natural and fundamental properties of Navier–Stokes (NS) equations and other mechanical principles, as we will see later.

It is accepted for a long time that Galilean invariance, which is a symmetry of Navier–Stokes equations, should be preserved by any turbulence model. Later, Speziale [2] was the first who recognized that the two-dimensional material indifference, which is another symmetry of the equations, is a key property that any subgrid-scale model should have. He has been followed by many authors such as Fureby and Tabor [3] and Wang [4,5].

Galilean invariance and the 2D-material indifference are two of the symmetry properties of NS equations. Indeed, NS equations have other transformations, called symmetries, which leave the equation and the set of solutions unchanged. Symmetries play an important role in turbulence. According to Nœther's theorem [6,7], each symmetry of a Lagrangian corresponds to a conservation law. For instance, in classical mechanics, if the Lagrangian has a time

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translation invariance property then the energy is conserved along the trajectory. Notice that, even if NS equations do not derive from a Lagrangian, it is possible to apply Nœther's theorem to them (see [8]), by extending the equations [9,10]. Next, symmetries permit to reduce the equations and to calculate self-similar solutions i.e. exact solutions which are unchanged when the symmetry transformation is applied [11]. Self-similar solutions constitute a powerful tool for the study of the equations. For example, Oberlack used them to derive classical and new turbulent scaling laws, such as algebraic and logarithmic laws few years earlier [12,13]. Such scaling laws was proved to be in a good agreement with experimental data and simulation results in many articles [12–15]. Next, Grassi et al. calculated self-similar solutions which represent vortex solutions of the NS equations [16]. Ünal used also this approach to show that the NS equations may admit solutions having the $5/3$ spectra of Kolmogorov [17]. Finally, notice that self-similar solutions give an information on the behaviour of the solutions at a very large time (see [18]).

To preserve the properties of the Navier–Stokes equations (self-similar solutions, conservation laws, ...) when approximating the solutions by RANS (Reynolds-Averaged Navier–Stokes) or LES (Large-Eddy Simulation) approach, the symmetries of the NS equations should not be violated by the turbulence model. Oberlack was the first who considered all the symmetries of Navier–Stokes equations when analyzing subgrid-scale models [19]. In his paper, he analysed a variety of standard models and concluded that many of them do not respect the invariance properties.

Beside symmetries, another property that turbulence models should admit is the consistency with the second law of thermodynamics. Unfortunately, many common models, such as the popular dynamic model [20], violate this principle, since they may induce a negative dissipation. Further, as it will be shown below, the conformity with this principle leads to the stability of the model.

The aim of this article is to expand the work of Oberlack in [19] by extending the symmetry analysis to a wider class of turbulence models, and next, to present a way of deriving new subgrid-scale models which, on one hand, preserve the symmetries of NS equations and, on the other hand, is conform to the second law of thermodynamics. This paper will be structured as follow. In Section 2, the symmetries of the Navier–Stokes equations will be revisited. This will be followed by a recall of some common models in Section 3. These models will then be analysed in Section 4 using the symmetry approach. In Section 5, a new class of subgrid models verifying the symmetry properties will be developed. Finally, in Section 6, this class will be restricted to models which are compatible with the second law of thermodynamics and it will be shown that this law leads to the stability of the new class of subgrid-scale models.

2. The symmetry group of Navier–Stokes equations

Consider a 3D incompressible Newtonian fluid, with density ρ and kinematic viscosity ν. The motion of this fluid is governed by Navier–Stokes equations

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} + \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) + \dfrac{1}{\rho}\nabla p = \operatorname{div} \mathsf{T}, \\ \operatorname{div} \boldsymbol{u} = 0, \end{cases} \tag{1}$$

where $\boldsymbol{u} = (u_i)_{i=1,2,3}$ and p are respectively the velocity and pressure fields and t the time variable. T is the tensor such that $\rho\mathsf{T}$ is the viscous constraint tensor. It can be linked to the strain rate tensor $\mathsf{S} = (\nabla \boldsymbol{u} + {}^{\mathrm{T}}\nabla \boldsymbol{u})/2$ according to the relation:

$$\mathsf{T} = \frac{\partial \psi}{\partial \mathsf{S}},$$

ψ being the positive and convex "potential" defined by:

$$\psi = \nu \operatorname{tr} \mathsf{S}^2.$$

Let us set some definition related to symmetry group theory. Let $\boldsymbol{y} = (t, \boldsymbol{x}, \boldsymbol{u}, p)$ where $\boldsymbol{x} = (x_i)_{i=1,2,3}$ is the space variable. To simplify, we designate (1) by

$$\mathcal{E}(\boldsymbol{y}) = 0. \tag{2}$$

A one-parameter transformation is a transformation

$$T_a : \boldsymbol{y} \mapsto \hat{\boldsymbol{y}} = \hat{\boldsymbol{y}}(\boldsymbol{y}, a)$$

which depends continuously on a real parameter a. T_a is called a symmetry of (2) if

$$\mathcal{E}(\boldsymbol{y}) = 0 \iff \mathcal{E}(\hat{\boldsymbol{y}}) = 0.$$

In other words, a one-parameter transformation is a symmetry of (2) if it leaves the equations unchanged (form invariant). In addition, it allows to map a given solution of (2) to another solution.

The set of the symmetries of (2) constitutes a (local) one-parameter group, called a *symmetry group* of (2). Assume that the neutral element of this group, the identity transformation, corresponds to $a = 0$ (in other words, the group is additive), then, the group is characterized by the variation of $\boldsymbol{y}$ under T_a around $a = 0$, which is represented by the following vector field or *infinitesimal generator*:

$$X = \frac{\partial \hat{\boldsymbol{y}}}{\partial a}\bigg|_{a=0} = \sum_i \xi_i \frac{\partial}{\partial y_i},$$

where $\boldsymbol{y} = (y_i)_i$ and $\xi_i = \partial \hat{y}_i / \partial a|_{a=0}$. More precisely, X is defined by:

$$X : f \in C^\infty \longmapsto Xf = \sum_i \xi_i \frac{\partial f}{\partial y_i}.$$

Knowing X, the expression of T_a can be calculated using the relations:

$$\frac{\partial \hat{\boldsymbol{y}}_i}{\partial a} = \xi_i(\hat{\boldsymbol{y}}) \quad \text{and} \quad \hat{\boldsymbol{y}}(a = 0) = \boldsymbol{y}.$$

Finally, it can be shown that the condition (2) is equivalent to (Lie's fundamental theorem, see [21]):

$$\mathcal{E}(\boldsymbol{y}) = 0 \iff X\mathcal{E}(\boldsymbol{y}) = 0.$$

Thanks to this last condition, Pukhnachev could calculate the infinitesimal generators of NS equations which are [22]:

$$\begin{aligned}
X_0 &= \frac{\partial}{\partial t}, \\
Y_0 &= \zeta(t)\frac{\partial}{\partial p}, \\
X_{ij} &= x_j \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_j} + u_j \frac{\partial}{\partial u_i} - u_i \frac{\partial}{\partial u_j}, \quad i = 1, 2,\ j > i, \\
X_i &= \alpha_i(t)\frac{\partial}{\partial x_i} + \dot{\alpha}_i(t)\frac{\partial}{\partial u_i} - \rho x_i \ddot{\alpha}_i(t)\frac{\partial}{\partial p}, \quad i = 1, 2, 3, \\
Y_1 &= 2t\frac{\partial}{\partial t} + \sum_{j=1}^{3}\left(x_j \frac{\partial}{\partial x_j} - u_j \frac{\partial}{\partial u_j}\right) - 2p\frac{\partial}{\partial p},
\end{aligned}$$

where ζ and $\boldsymbol{\alpha} = (\alpha_i)_i$ are arbitrary functions and the symbol dot ($\dot{}$) stands for derivative.

Until now, we considered symmetries which act on the variables t, $\boldsymbol{x}$, $\boldsymbol{u}$ and p. It is also possible to consider symmetries which, in addition to these variables, act on the parameter ν of the flow, i.e. having the following form:

$$(t, \boldsymbol{x}, \boldsymbol{u}, p, \nu) \mapsto (\hat{t}, \hat{\boldsymbol{x}}, \hat{\boldsymbol{u}}, \hat{p}, \hat{\nu}).$$

Such a symmetry, called *equivalence transformation*, takes a solution of the NS equations into a new solution of the NS equations with a different value of ν. Doing so, we obtain an additional infinitesimal generator [23]:

$$Y_2 = \sum_{j=1}^{3} x_j \frac{\partial}{\partial x_j} + \sum_{j=1}^{3} u_j \frac{\partial}{\partial u_j} + 2p\frac{\partial}{\partial p} + 2\nu\frac{\partial}{\partial \nu}.$$

From the above generators, the symmetry group of the NS equations can be obtained. It is spanned by the following symmetries:

- the *time translation* which corresponds to X_0:

$$(t, \boldsymbol{x}, \boldsymbol{u}, p) \mapsto (t + a, \boldsymbol{x}, \boldsymbol{u}, p), \tag{3}$$

- the *pressure translation* generated by Y_0:

$$(t,\boldsymbol{x},\boldsymbol{u},p)\mapsto\bigl(t,\boldsymbol{x},\boldsymbol{u},p+\zeta(t)\bigr),\tag{4}$$

- the *rotation* corresponding to the three X_{ij}'s:

$$(t,\boldsymbol{x},\boldsymbol{u},p)\mapsto(t,\mathsf{R}\boldsymbol{x},\mathsf{R}\boldsymbol{u},p),\tag{5}$$

- the *generalized Galilean transformation* characterized by the three X_i's:

$$(t,\boldsymbol{x},\boldsymbol{u},p)\mapsto\left(t,\boldsymbol{x}+\boldsymbol{\alpha}(t),\boldsymbol{u}+\dot{\boldsymbol{\alpha}}(t),p-\rho\boldsymbol{x}\cdot\ddot{\boldsymbol{\alpha}}(t)-\frac{1}{2}\rho\boldsymbol{\alpha}(t)\cdot\ddot{\boldsymbol{\alpha}}(t)\right),\tag{6}$$

- the *first scaling transformation* corresponding to Y_1:

$$(t,\boldsymbol{x},\boldsymbol{u},p)\mapsto\bigl(\mathrm{e}^{2a}t,\mathrm{e}^{a}\boldsymbol{x},\mathrm{e}^{-a}\boldsymbol{u},\mathrm{e}^{-2a}p\bigr),\tag{7}$$

- and the *second scaling transformation* corresponding to Y_2:

$$(t,\boldsymbol{x},\boldsymbol{u},p)\mapsto\bigl(t,\mathrm{e}^{a}\boldsymbol{x},\mathrm{e}^{a}\boldsymbol{u},\mathrm{e}^{2a}p,\mathrm{e}^{2a}\nu\bigr).\tag{8}$$

In these expressions, R is a constant rotation matrix, i.e. $\mathsf{R}^{\mathrm{T}}\mathsf{R}=\mathsf{I_d}$ and $\det\mathsf{R}=1$, $\mathsf{I_d}$ being the identity matrix. Note that space translations can be obtained from (6) by choosing $\boldsymbol{\alpha}$ constant and the classical Galilean transformation by choosing $\boldsymbol{\alpha}$ linear in t. The first scaling transformation (7) shows how the velocity and the pressure are affected when the spatio-temporal scale is multiplied by $(\mathrm{e}^a,\mathrm{e}^{2a})$, and the second scaling transformation (8) shows the consequence on the velocity and the pressure of a spatial scale change.

The method above of calculating the symmetries is an application of Lie theory (see [7] or [21] for a more rigorous and more complete presentation). The NS equations admit other known symmetries which are not one-parameter symmetries and which could not be calculated by the same method. They are

- the *reflection*:

$$(t,\boldsymbol{x},\boldsymbol{u},p)\mapsto(t,\Lambda\boldsymbol{x},\Lambda\boldsymbol{u},p),\tag{9}$$

  where

$$\Lambda=\begin{pmatrix}\iota_1&0&0\\0&\iota_2&0\\0&0&\iota_3\end{pmatrix}\quad\text{with }\iota_i=\pm1,\ i=1,2,3,$$

- and the *material indifference* in the limit of a 2D flow in a simply connected domain [24]:

$$(t,\boldsymbol{x},\boldsymbol{u},p)\mapsto(t,\hat{\boldsymbol{x}},\hat{\boldsymbol{u}},\hat{p}),\tag{10}$$

  with

$$\hat{\boldsymbol{x}}=\mathsf{R}(t)\boldsymbol{x},\qquad\hat{\boldsymbol{u}}=\mathsf{R}(t)\boldsymbol{u}+\dot{\mathsf{R}}(t)\boldsymbol{x},\qquad\hat{p}=p-3\omega\varphi+\frac{1}{2}\omega^2\|\boldsymbol{x}\|^2,$$

  where $\mathsf{R}(t)$ is a 2D rotation matrix with angle ωt, ω an arbitrary real constant, φ the usual 2D stream function defined by:

$$\boldsymbol{u}=\mathrm{curl}(\varphi\boldsymbol{e}_3),$$

  $\boldsymbol{e}_3$ the unit vector perpendicular to the plane of the flow and $\|\cdot\|$ indicates the Euclidean norm.

Notice that other kinds of symmetry exist, such as symmetries (called Lie Bäcklund or generalized symmetry transformations, [7,21]) which explicitly include the derivatives of the dependent variables $\boldsymbol{u}$ and p. They are out of the consideration of this article.

The symmetries have an important role, as seen in the introduction. In some extent, they contains the physics of the equations. So, turbulent models have to respect them, otherwise fundamental properties of the flow are lost during the modelling.

In the next section, we remind some standard models using large-eddy simulation approach, in order to analyse them later according to the respect of the symmetries.

3. Common subgrid models

Large-eddy simulation method consists in reducing the computation time by dropping the small scales of the unknowns. This is done by using a filter. The large or resolved scales $\bar{\phi}$ of a quantity ϕ are defined by the convolution:

$$\bar{\phi} = G * \phi,$$

where G is the filter kernel and the small scales ϕ' are given by

$$\phi' = \phi - \bar{\phi}.$$

The velocity and the pressure $(\boldsymbol{u}, p)$ are then approximated by $(\bar{\boldsymbol{u}}, \bar{p})$. To obtain $(\bar{\boldsymbol{u}}, \bar{p})$, Eqs. (1) are filtered. This leads to:

$$\begin{cases} \dfrac{\partial \bar{\boldsymbol{u}}}{\partial t} + \operatorname{div}(\bar{\boldsymbol{u}} \otimes \bar{\boldsymbol{u}}) + \dfrac{1}{\rho}\nabla \bar{p} = \operatorname{div}\big(\overline{\mathsf{T}} + \mathsf{T}_s\big), \\ \operatorname{div} \bar{u} = 0, \end{cases} \tag{11}$$

where T_s is the subgrid stress tensor defined by $\mathsf{T}_s = \bar{\boldsymbol{u}} \otimes \bar{\boldsymbol{u}} - \overline{\boldsymbol{u} \otimes \boldsymbol{u}}$ which must be modeled to close the equations. Currently, a very large number of models exists. Some of the most common ones will be reminded here. They will be classified in four categories: turbulent viscosity, gradient-type, similarity-type and Lund–Novikov-type models.

3.1. Turbulent viscosity models

Turbulent viscosity models are models which can be written in the following form:

$$\mathsf{T}_s^d = \nu_s \bar{\mathsf{S}},$$

where the superscript $(^d)$ represents the deviatoric part of a tensor and is defined by:

$$\mathsf{Q} \mapsto \mathsf{Q}^d = \mathsf{Q} - \frac{1}{3}(\operatorname{tr}\mathsf{Q})\mathsf{I_d}$$

and ν_s is the turbulent viscosity.

Smagorinsky model (see [1]) is one of the most widely used models. It uses the local equilibrium hypothesis to calculate the turbulent viscosity. It has the following expression:

$$\mathsf{T}_s^d = (C_S\bar{\delta})^2 |\bar{\mathsf{S}}| \bar{\mathsf{S}}, \tag{12}$$

where $C_S \simeq 0.148$ is the Smagorinsky constant, $\bar{\delta}$ the filter width and $|\bar{\mathsf{S}}| = \sqrt{2\operatorname{tr}(\bar{\mathsf{S}}^2)}$.

In order to reduce the modelling error of the Smagorinsky model, Lilly ([20]) proposes a dynamical evaluation of the constant C_S by a least square approach. This leads to the *dynamic model*:

$$\mathsf{T}_s^d = C_d \bar{\delta}^2 |\bar{\mathsf{S}}| \bar{\mathsf{S}}, \quad \text{with } C_d = \frac{\operatorname{tr}(\mathsf{LM})}{\operatorname{tr}\mathsf{M}^2}. \tag{13}$$

In these terms,

$$\mathsf{L} = \tilde{\bar{\boldsymbol{u}}} \otimes \tilde{\bar{\boldsymbol{u}}} - \widetilde{\bar{\boldsymbol{u}} \otimes \bar{\boldsymbol{u}}}, \qquad \mathsf{M} = \bar{\delta}^2 \widetilde{|\bar{\mathsf{S}}|\bar{\mathsf{S}}} - \tilde{\bar{\delta}}^2 |\tilde{\bar{\mathsf{S}}}| \tilde{\bar{\mathsf{S}}},$$

where the tilde $(\tilde{\ })$ symbolizes a test filter, with a filter width $\tilde{\bar{\delta}}$.

The last turbulent viscosity model which will be considered is the structure function model.

Métais and Lesieur [25] make the hypothesis that the turbulent viscosity depends on the energy at the cutoff, knowing that there is a relation between the energy density in Fourier space and the second order structure function

$$f_2(\boldsymbol{x}, r) = \int_{\|z\|=r} \big\|\boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{u}(\boldsymbol{x}+z)\big\|^2 \,\mathrm{d}z.$$

Using a dimensional argument and Kolmogorov theory, they then deduce the turbulent viscosity and propose the *structure function model*:

$$\mathsf{T}_s^d = C_{SF}\bar{\delta}\sqrt{\overline{F}_2(\bar{\delta})}\,\bar{\mathsf{S}}, \tag{14}$$

where $\overline{F}_2$ is the function

$$r \mapsto \int_{\mathbb{R}^3} \bar{f}_2(\boldsymbol{x}, r)\,\mathrm{d}\boldsymbol{x}$$

and C_{SF} is the constant of the model.

The next category of models which will be reminded is composed by the gradient-type models.

3.2. Gradient-type models

To establish the gradient-type models, one decomposes the subgrid stress tensor as follow:

$$\mathsf{T}_s = \bar{\boldsymbol{u}} \otimes \bar{\boldsymbol{u}} - \left(\overline{\bar{\boldsymbol{u}} \otimes \bar{\boldsymbol{u}}} + \overline{\bar{\boldsymbol{u}} \otimes \boldsymbol{u}'} + \overline{\boldsymbol{u}' \otimes \bar{\boldsymbol{u}}} + \overline{\boldsymbol{u}' \otimes \boldsymbol{u}'}\right).$$

Next, each term between the brackets are written in Fourier space. Then, the Fourier transform of the filter, which is assumed to be Gaussian, is approximated by an appropriate function. Finally, the inverse Fourier transform is computed. The models in this category differ by the way in which the Fourier transform of the filter is approximated.

The simplest approximation of the Fourier transform of the filter is obtained by Taylor series expansions according to the filter width $\bar{\delta}$:

$$\mathcal{F}(G)(k) = 1 - \frac{\|k\|^2}{24}\bar{\delta}^2 + \mathrm{O}\left(\bar{\delta}^4\right).$$

Neglecting terms in $\mathrm{O}(\bar{\delta}^4)$ and returning to physical space, one gets the *gradient model* [26]:

$$\mathsf{T}_s = -\frac{\bar{\delta}^2}{12}\nabla\bar{\boldsymbol{u}}^{\mathrm{T}}\nabla\bar{\boldsymbol{u}}. \tag{15}$$

The gradient model is not dissipative enough and not stable [27,28]. Thus, it is generally combined with the Smagorinsky model. This gives the *Taylor model*:

$$\mathsf{T}_s = -\frac{\bar{\delta}^2}{12}\nabla\bar{\boldsymbol{u}}^{\mathrm{T}}\nabla\bar{\boldsymbol{u}} + C_g\bar{\delta}^2\left|\bar{\mathsf{S}}\right|\bar{\mathsf{S}} \tag{16}$$

C_g being a constant.

In fact, the Taylor approximation of the Fourier transform of the filter tends to accentuate the small frequencies instead of attenuating them. To avoid this accentuation, a rational approximation is proposed [29,30]:

$$\mathcal{F}(G)(k) = \frac{1}{1 + (\|k\|^2/24)\bar{\delta}^2} + \mathrm{O}\left(\bar{\delta}^4\right).$$

After returning to the physical space, one gets:

$$\mathsf{T}_s = -\frac{\bar{\delta}^2}{12}\left(\mathsf{I_d} - \frac{\bar{\delta}^2}{24}\nabla^2\right)^{-1}\left[\nabla\bar{\boldsymbol{u}}^{\mathrm{T}}\nabla\bar{\boldsymbol{u}}\right] + C_g\bar{\delta}^2\left|\bar{\mathsf{S}}\right|\bar{\mathsf{S}}.$$

To avoid the inversion of the operator $(\mathsf{I_d} - \frac{\bar{\delta}^2}{24}\nabla^2)$, T_s is approximated by:

$$\mathsf{T}_s = -\frac{\bar{\delta}^2}{12}G * \left[\nabla\bar{\boldsymbol{u}}^{\mathrm{T}}\nabla\bar{\boldsymbol{u}}\right] + C_g\bar{\delta}^2\left|\bar{\mathsf{S}}\right|\bar{\mathsf{S}}. \tag{17}$$

The convolution is done numerically. The model (17) is called the *rational model*.

3.3. Similarity-type models

Models of this category are based on the hypothesis that the statistical structure of the small scales are similar to the statistical structure of the smallest resolved scales. The separation of the resolved scales is done using a test filter (symbolized by $\tilde{\ }$). The largest resolved scales are then represented by $\tilde{\bar{\boldsymbol{u}}}$ and the smallest ones by $\bar{\boldsymbol{u}} - \tilde{\bar{\boldsymbol{u}}}$. From this hypothesis, we deduce the *similarity model*:

$$\mathsf{T}_s = \tilde{\bar{\boldsymbol{u}}} \otimes \tilde{\bar{\boldsymbol{u}}} - \widetilde{\bar{\boldsymbol{u}} \otimes \bar{\boldsymbol{u}}}. \tag{18}$$

From (18), many other models can be obtained by multiplying by a coefficient, by filtering again the whole expression or by mixing with a Smagorinsky-type model.

The last models that we will consider are ones which derive from the Lund and Novikov's model.

3.4. Lund–Novikov-type models

Lund and Novikov include the filtered vorticity tensor $\overline{\mathsf{W}} = (\nabla\bar{\boldsymbol{u}} - {}^{\mathrm{T}}\nabla\bar{\boldsymbol{u}})$ in the expression of the subgrid model:

$$\mathsf{T}_s = \mathsf{T}_s(\bar{\mathsf{S}}, \overline{\mathsf{W}}).$$

Cayley–Hamilton theorem gives then the *Lund–Novikov model* (see [1]):

$$-\mathsf{T}_s^d = C_1\bar{\delta}^2|\bar{\mathsf{S}}|\bar{\mathsf{S}} + C_2\bar{\delta}^2(\bar{\mathsf{S}}^2)^d + C_3\bar{\delta}^2(\overline{\mathsf{W}}^2)^d + C_4\bar{\delta}^2(\bar{\mathsf{S}}\overline{\mathsf{W}} - \overline{\mathsf{W}}\bar{\mathsf{S}}) + C_5\bar{\delta}^2\frac{1}{|\bar{\mathsf{S}}|}(\bar{\mathsf{S}}^2\overline{\mathsf{W}} - \bar{\mathsf{S}}\overline{\mathsf{W}}^2), \tag{19}$$

where the coefficients C_i depend on the invariants obtained from $\bar{\mathsf{S}}$ and $\overline{\mathsf{W}}$. The expression of these coefficients are so complex that they are considered as constants and evaluated with statistic techniques.

To reduce the computational cost of the previous model, Kosovic suggests a simplification and proposes the following model:

$$-\mathsf{T}_s^d = (C_0\bar{\delta})^2\left[2|\bar{\mathsf{S}}|\bar{\mathsf{S}} + C_1(\bar{\mathsf{S}}^2)^d + C_2(\bar{\mathsf{S}}\overline{\mathsf{W}} - \overline{\mathsf{W}}\bar{\mathsf{S}})\right], \tag{20}$$

where the constants C_0, C_1 and C_2 are calculated using the theory of homogeneous and isotropic turbulence.

Many other types of models exist. In this article, we only deal with ones which do not introduce new transport equations.

The analysis of the models presented above is the object of the next section.

4. Model analysis

The NS equations (1) is preserved by each of transformations (3)–(10). We then require that the filtered equations (11) is also preserved by all of these transformations, since the solution $(\bar{\boldsymbol{u}}, \bar{p})$ of the filtered equations is expected to be a good approximation of $(\boldsymbol{u}, p)$. More clearly, if a transformation

$$T : (t, \boldsymbol{x}, \boldsymbol{u}, p) \mapsto (\hat{t}, \hat{\boldsymbol{x}}, \hat{\boldsymbol{u}}, \hat{p})$$

is a symmetry of NS equations, we require that the model is such that the same transformation, applied to the filtered quantities:

$$T : (t, \boldsymbol{x}, \bar{\boldsymbol{u}}, \bar{p}) \mapsto (\hat{t}, \hat{\boldsymbol{x}}, \hat{\bar{\boldsymbol{u}}}, \hat{\bar{p}})$$

is a symmetry of the filtered equations (11). When this constraint holds, the model will be said to be *invariant* under the symmetry. If the model is invariant under all the symmetries of (1), we may expect to preserve certain properties of the NS equations, such as conservation laws, wall laws, exact solutions, spectrum properties, ... when approximating $(\boldsymbol{u}, p)$ by $(\bar{\boldsymbol{u}}, \bar{p})$.

The symmetries of the filtered equations (11) may be different than (3)–(10). As was first suggested by Speziale [31] and later extended by Oberlack [19], we require that each symmetry of (1) remains a symmetry of (11), or, in other words, that the symmetry group of (1) is included in the symmetry group of (11).

We will make the hypothesis that the test filters do not destroy the symmetry properties, i.e.

$$\hat{\tilde{\phi}} = \tilde{\hat{\phi}}$$

for any quantity ϕ.

For the subsequent analysis, the symmetries of (1) will be gathered in four categories:

- the translations, containing time translation, pressure translation and the generalized Galilean transformation,
- the rotation and the reflection,
- the two scaling transformations,
- and the 2D material indifference.

The aim is to search for those models which are invariant under the symmetries within the considered category.

4.1. Invariance under the translations

Since almost all existing models are autonomous in time and pressure, (11) remains unchanged when the time translation (3) or the pressure translation (4) is applied. Almost all models are then invariant under (3) and (4).

Let us now consider the generalized Galilean transformation (6), applied to the filtered variables:

$$(t, \boldsymbol{x}, \bar{\boldsymbol{u}}, \bar{p}) \mapsto \left(t, \boldsymbol{x} + \boldsymbol{\alpha}(t), \bar{\boldsymbol{u}} + \dot{\boldsymbol{\alpha}}(t), \bar{p} - \rho \boldsymbol{x} \cdot \ddot{\boldsymbol{\alpha}}(t) - \frac{1}{2}\rho \boldsymbol{\alpha}(t) \cdot \ddot{\boldsymbol{\alpha}}(t)\right).$$

All models in Section 2, in which $\boldsymbol{x}$ and $\bar{\boldsymbol{u}}$ are present only through $\nabla\bar{\boldsymbol{u}}$ are invariant since

$$\widehat{\nabla}\hat{\bar{\boldsymbol{u}}} = \nabla\hat{\bar{\boldsymbol{u}}} = \nabla\bar{\boldsymbol{u}},$$

where $\widehat{\nabla} = {}^{\mathrm{T}}(\partial/\partial\hat{x}_1, \partial/\partial\hat{x}_2, \partial/\partial\hat{x}_3)$.

The remaining models, i.e. the dynamic and the similarity models are also invariant because

$$\widetilde{(\bar{\boldsymbol{u}} + \dot{\boldsymbol{\alpha}}) \otimes (\bar{\boldsymbol{u}} + \dot{\boldsymbol{\alpha}})} - \widetilde{(\bar{\boldsymbol{u}} + \dot{\boldsymbol{\alpha}})} \otimes \widetilde{(\bar{\boldsymbol{u}} + \dot{\boldsymbol{\alpha}})} = \widetilde{\bar{\boldsymbol{u}} \otimes \bar{\boldsymbol{u}}} - \tilde{\bar{\boldsymbol{u}}} \otimes \tilde{\bar{\boldsymbol{u}}}.$$

4.2. Invariance under the rotation and reflection

The transformations (5) and (9) can be put together in a transformation:

$$(t, \boldsymbol{x}, \boldsymbol{u}, p) \mapsto (t, \Upsilon\boldsymbol{x}, \Upsilon\boldsymbol{u}, p),$$

where Υ is a (constant) rotation or reflection matrix. This transformation, when applied to the filtered variables, is a symmetry of (11) if and only if

$$\widehat{\mathsf{T}}_s = \Upsilon \mathsf{T}_s {}^{\mathrm{T}}\Upsilon. \tag{21}$$

For the Smagorinsky model, one has:

$$\widehat{\nabla}\hat{\bar{\boldsymbol{u}}} = \left[\nabla\left(\hat{\bar{\boldsymbol{u}}}\right)\right]^{\mathrm{T}}\Upsilon = \left[\nabla(\Upsilon\bar{\boldsymbol{u}})\right]^{\mathrm{T}}\Upsilon = \Upsilon[\nabla\bar{\boldsymbol{u}}]^{\mathrm{T}}\Upsilon. \tag{22}$$

This leads to the objectivity of $\bar{\mathsf{S}}$:

$$\widehat{\bar{\mathsf{S}}} = \Upsilon\bar{\mathsf{S}}{}^{\mathrm{T}}\Upsilon.$$

And since $|\widehat{\bar{\mathsf{S}}}| = |\bar{\mathsf{S}}|$, (21) is verified. The Smagorinsky model is then invariant under reflection and rotation.

From the similarity model (18), one has:

$$\hat{\bar{\boldsymbol{u}}} \otimes \hat{\bar{\boldsymbol{u}}} = (\Upsilon\bar{\boldsymbol{u}}) \otimes (\Upsilon\bar{\boldsymbol{u}}) = \Upsilon(\bar{\boldsymbol{u}} \otimes \bar{\boldsymbol{u}})'{}^{\mathrm{T}}\Upsilon.$$

Using these relations, the invariance of the similarity model can easily been deduced.

The same relations are sufficient to prove the invariance of the dynamic model (13) since the trace remains invariant under a change of basis.

The structure function model (14) is invariant because the function $\overline{F}_2$ does not change under a rotation or a reflection.

Relations (22) can be used again to prove the invariance of all of the gradient-type models.

Finally, since

$$\widehat{\overline{\mathsf{W}}} = \Upsilon\overline{\mathsf{W}}{}^{\mathrm{T}}\Upsilon,$$

the Lund–Novikov-type models are also invariant.

As a result, any model of Section 2 is then invariant under rotation and reflection.

4.3. Invariance under the scaling transformations

The two scaling transformations (7) and (8) can be combined into the following two-parameter scaling transformation:

$$(t, \boldsymbol{x}, \boldsymbol{u}, p, \nu) \mapsto \left(\mathrm{e}^{2a}t, \mathrm{e}^{b+a}\boldsymbol{x}, \mathrm{e}^{b-a}\boldsymbol{u}, \mathrm{e}^{2b-2a}p, \mathrm{e}^{2b}\nu\right). \tag{23}$$

The first scaling transformation (7) corresponds to $b = 0$ and the second, (8), to $a = 0$. It can be verified that (11) is invariant under the scaling transformation (23) if and only if

$$\widehat{\mathsf{T}}_s = \mathrm{e}^{2b-2a}\mathsf{T}_s. \tag{24}$$

From (23) one may deduce that $\widehat{\bar{\mathsf{S}}} = \mathrm{e}^{-2a}\bar{\mathsf{S}}$ and hence, condition (24) is equivalent to:

$$\hat{\nu}_s = \mathrm{e}^{2b}\nu_s \tag{25}$$

for any turbulent viscosity model.

For the most common subgrid-scale model, the Smagorinsky model, we have:

$$\hat{\nu}_s = C_S\bar{\delta}^2|\widehat{\bar{\mathsf{S}}}| = \mathrm{e}^{-2a}C_S\bar{\delta}^2|\bar{\mathsf{S}}| = \mathrm{e}^{-2a}\nu_s.$$

The condition (25) is violated as well for $a = 0$ as for $b = 0$. The model is invariant neither under the first nor the second scaling transformation. Note that the filter width $\bar{\delta}$ does not vary since it is an external length scale and has no functional dependence on the variables of the flow. This is one of the results of Oberlack in [19]. Further therein, it is shown that the dynamic procedure used in (13) restores the scaling invariance. Indeed, it can be shown that:

$$\widehat{C}_d = \mathrm{e}^{2b+2a}C_d.$$

Thus:

$$\hat{\nu}_s = \widehat{C}_d\bar{\delta}^2|\widehat{\bar{\mathsf{S}}}| = C_d\bar{\delta}^2|\bar{\mathsf{S}}| = \mathrm{e}^{2b}\nu_s.$$

The dynamic model is then invariant under the scaling transformations.

For the structure function model, we have:

$$\widehat{\overline{F}}_2 = \mathrm{e}^{2b-2a}\overline{F}_2$$

and then

$$\hat{\nu}_s = \mathrm{e}^{b-a}\nu_{sm},$$

which proves that the model is not invariant under (23).

Further, since

$$\widehat{\nabla}\hat{\bar{\boldsymbol{u}}} = \mathrm{e}^{-2a}\nabla\bar{\boldsymbol{u}}$$

the gradient model (15) violates (24), T_s varying in the following way:

$$\widehat{\mathsf{T}}_s = \mathrm{e}^{-4a}\mathsf{T}_s.$$

This also implies that none of the gradient-type models is invariant under the scaling transformations.

Next, it is straight forward to show that the similarity model (18) verifies (24) and hence is invariant.

Lastly, the Lund–Novikov-type models are not invariant, since they contain a term similar to the Smagorinsky model, which is not invariant under (23).

In fact, in [19], Oberlack proved that no model where the external length scale $\bar{\delta}$ appears explicitly is invariant under the scaling transformations. Note that the dynamic model can be written in the following form:

$$\mathsf{T}_s^d = \frac{\mathrm{tr}(\mathsf{LN})}{\mathrm{tr}(\mathsf{N}^2)}|\bar{\mathsf{S}}|\bar{\mathsf{S}},$$

where

$$\mathsf{N} = |\widetilde{\bar{\mathsf{S}}|\bar{\mathsf{S}}} - (\tilde{\bar{\delta}}/\bar{\delta})^2|\tilde{\bar{\mathsf{S}}}|\tilde{\bar{\mathsf{S}}}.$$

It is then the ratio $\hat{\bar{\delta}}/\bar{\delta}$ which is present in the model but neither $\bar{\delta}$ alone nor $\hat{\bar{\delta}}$ alone.

In summary, the dynamic and the similarity models are the only invariant models under the scaling transformations. Though the scaling transformations have a particular importance, as mentioned in the introduction. One other interesting point is that, unlike the similarity model (18), the gradient model (15) is not invariant under the scaling transformations. Nevertheless, the gradient model can formally be deduced from the similarity model using a Taylor approximation [1]. A Taylor approximation may then destroy a symmetry property.

The consequence of the breaking of the scaling symmetries will be discussed later.

The last symmetry of the NS equations to be considered is the material indifference (10), in the limit of 2D flow, in a connected domain.

4.4. Invariance under the 2D material indifference

The 2D material indifference (10) corresponds to a time-dependent plane rotation, with a compensation in the pressure term. We will not write explicitly the dependence on time of the rotation matrix R. A model is invariant if

$$\widehat{\mathsf{T}}_s = \mathsf{R}\mathsf{T}_s{}^{\mathrm{T}}\mathsf{R}.$$

The objectivity of $\bar{\mathsf{S}}$ (see Subsection 4.2) directly leads to the invariance of the Smagorinsky model.

For the similarity model (18), we have:

$$\widehat{\mathsf{T}}_s = \mathsf{R}\mathsf{T}_s{}^{\mathrm{T}}\mathsf{R} + \mathsf{R}\big(\widetilde{\bar{\boldsymbol{u}} \otimes \boldsymbol{x}} - \tilde{\bar{\boldsymbol{u}}} \otimes \tilde{\boldsymbol{x}}\big)^{\mathrm{T}}\mathsf{R} + \dot{\mathsf{R}}\big(\widetilde{\boldsymbol{x} \otimes \bar{\boldsymbol{u}}} - \tilde{\boldsymbol{x}} \otimes \tilde{\bar{\boldsymbol{u}}}\big)^{\mathrm{T}}\mathsf{R} + \dot{\mathsf{R}}\big(\widetilde{\boldsymbol{x} \otimes \boldsymbol{x}} - \tilde{\boldsymbol{x}} \otimes \tilde{\boldsymbol{x}}\big)^{\mathrm{T}}\dot{\mathsf{R}}.$$

Consequently, if the test filter is such that

$$\big(\widetilde{\bar{\boldsymbol{u}} \otimes \boldsymbol{x}} - \tilde{\bar{\boldsymbol{u}}} \otimes \tilde{\boldsymbol{x}}\big) = 0, \qquad \big(\widetilde{\boldsymbol{x} \otimes \bar{\boldsymbol{u}}} - \tilde{\boldsymbol{x}} \otimes \tilde{\bar{\boldsymbol{u}}}\big) = 0, \qquad \big(\widetilde{\boldsymbol{x} \otimes \boldsymbol{x}} - \tilde{\boldsymbol{x}} \otimes \tilde{\boldsymbol{x}}\big) = 0, \tag{26}$$

then the similarity model is invariant under the 2D material indifference. All filters do not have this property. For instance, it can be shown (see [8]) that, for the usual box filter, these terms are respectively in the error order $\mathrm{O}(\tilde{\bar{\delta}})$, $\mathrm{O}(\tilde{\bar{\delta}})$ and $\mathrm{O}(\tilde{\bar{\delta}}^2)$.

Under the same conditions (26) on the test filter, the dynamic model (13) is also invariant.

Next, the structure function model (14) is invariant if and only if

$$\widehat{\overline{F}}_2 = \overline{F}_2. \tag{27}$$

Let us calculate $\widehat{\overline{F}}_2$. Let $\bar{\boldsymbol{u}}_z$ be the function $\boldsymbol{x} \mapsto \bar{\boldsymbol{u}}(\boldsymbol{x}+\boldsymbol{z})$. Then

$$\widehat{\overline{F}}_2 = \int\limits_{\|z\|=\bar{\delta}} \big\|(\mathsf{R}\bar{\boldsymbol{u}} + \dot{\mathsf{R}}\boldsymbol{x}) - (\mathsf{R}\boldsymbol{u}_z + \dot{\mathsf{R}}\boldsymbol{x} + \dot{\mathsf{R}}\boldsymbol{z})\big\|^2 \,\mathrm{d}z = \int\limits_{\|z\|=\bar{\delta}} \big\|\bar{\boldsymbol{u}} - \bar{\boldsymbol{u}}_z - {}^{\mathrm{T}}\mathsf{R}\dot{\mathsf{R}}\boldsymbol{z}\big\| \,\mathrm{d}z.$$

Knowing that ${}^{\mathrm{T}}\mathsf{R}\dot{\mathsf{R}}\boldsymbol{z} = \omega\boldsymbol{e}_3 \times \boldsymbol{z}$, we obtain:

$$\widehat{\overline{F}}_2 = \overline{F}_2 + 2\pi\omega^2\bar{\delta}^3 - 2\omega \int\limits_{\|z\|=\bar{\delta}} (\bar{\boldsymbol{u}} - \bar{\boldsymbol{u}}_z)\cdot(\boldsymbol{e}_3 \times \boldsymbol{z})\,\mathrm{d}z.$$

Condition (27) is violated. So, the structure function model is not invariant under the 2D material indifference.

For the gradient model, we have:

$$\widehat{\nabla}\hat{\bar{\boldsymbol{u}}} = \mathsf{R}\nabla\bar{\boldsymbol{u}}{}^{\mathrm{T}}\mathsf{R} + \dot{\mathsf{R}}{}^{\mathrm{T}}\mathsf{R}, \qquad {}^{\mathrm{T}}\widehat{\nabla}\hat{\bar{\boldsymbol{u}}} = \mathsf{R}\,{}^{\mathrm{T}}\nabla\bar{\boldsymbol{u}}\,{}^{\mathrm{T}}\mathsf{R} + \mathsf{R}\,{}^{\mathrm{T}}\dot{\mathsf{R}}. \tag{28}$$

Let J be the matrix such that $\dot{\mathsf{R}}{}^{\mathrm{T}}\mathsf{R} = -\omega\mathsf{J} = -\mathsf{R}\,{}^{\mathrm{T}}\dot{\mathsf{R}}$ or, in a more explicit form:

$$\mathsf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then,

$$\big(\widehat{\nabla\boldsymbol{u}^{\mathrm{T}}\nabla\boldsymbol{u}}\big) = \mathsf{R}\big(\nabla\boldsymbol{u}^{\mathrm{T}}\nabla\boldsymbol{u}\big)^{\mathrm{T}}\mathsf{R} + \omega\mathsf{R}\nabla\bar{\boldsymbol{u}}\,{}^{\mathrm{T}}\mathsf{R}\mathsf{J} - \omega\mathsf{J}\mathsf{R}\,{}^{\mathrm{T}}\nabla\bar{\boldsymbol{u}}{}^{\mathrm{T}}\mathsf{R} + \omega^2\mathsf{I}_{\mathsf{d}}.$$

The commutativity between J and R finally leads to:

$$\widehat{\mathsf{T}}_s = \mathsf{R}\mathsf{T}_s{}^{\mathrm{T}}\mathsf{R} + \omega\mathsf{R}\big(\nabla\bar{\boldsymbol{u}}\mathsf{J} - \mathsf{J}^{\mathrm{T}}\nabla\bar{\boldsymbol{u}}\big) + \omega^2\mathsf{I}_{\mathsf{d}}.$$

This proves that the gradient model is not invariant under the 2D material indifference.

The other gradient-type models inherit the lack of invariance of the gradient model.

It remains the Lund–Novikov-type models to be investigated. We will begin with the Kosovic model (20) since it is simpler. The first two terms of (20) are invariant. For the vorticity tensor, it follows from (28) that:

$$\widehat{\overline{\mathsf{W}}} = \mathsf{R}\overline{\mathsf{W}}{}^{\mathrm{T}}\mathsf{R} - \omega\mathsf{J}.$$

Thus,

$$\widehat{\bar{\mathsf{S}}}\widehat{\overline{\mathsf{W}}} - \widehat{\overline{\mathsf{W}}}\widehat{\bar{\mathsf{S}}} = \mathsf{R}\big(\bar{\mathsf{S}}\overline{\mathsf{W}} - \overline{\mathsf{W}}\bar{\mathsf{S}}\big)^{\mathrm{T}}\mathsf{R} - \omega\mathsf{R}\big(\bar{\mathsf{S}}\mathsf{J} - \mathsf{J}\bar{\mathsf{S}}\big)^{\mathrm{T}}\mathsf{R}$$

using again the commutativity between J and R. As for them, $\bar{\mathsf{S}}$ and J are not commutative. In fact, using properties of $\bar{\mathsf{S}}$, it can be shown that $\bar{\mathsf{S}}\mathsf{J} = -\mathsf{J}\bar{\mathsf{S}}$. This implies that

$$\widehat{\bar{\mathsf{S}}}\widehat{\overline{\mathsf{W}}} - \widehat{\overline{\mathsf{W}}}\widehat{\bar{\mathsf{S}}} = \mathsf{R}\big(\bar{\mathsf{S}}\overline{\mathsf{W}} - \overline{\mathsf{W}}\bar{\mathsf{S}}\big)^{\mathrm{T}}\mathsf{R} - 2\omega\mathsf{R}\bar{\mathsf{S}}\mathsf{J}\,^{\mathrm{T}}\mathsf{R} \tag{29}$$

which shows that Kosovic model is not invariant under the 2D material indifference.

Lastly, we consider the Lund–Novikov model (19). We have:

$$\widehat{\overline{\mathsf{W}}}^2 = \mathsf{R}\overline{\mathsf{W}}^2\,^{\mathrm{T}}\mathsf{R} - \omega\mathsf{R}\big(\mathsf{J}\overline{\mathsf{W}} + \overline{\mathsf{W}}\mathsf{J}\big)^{\mathrm{T}}\mathsf{R} - \omega^2\mathsf{I_d}.$$

Since $\overline{\mathsf{W}}$ is anti-symmetric and the flow is 2D, $\overline{\mathsf{W}}$ is in the form:

$$\overline{\mathsf{W}} = \begin{pmatrix} 0 & \overline{w} \\ -\overline{w} & 0 \end{pmatrix}.$$

A direct calculation leads then to

$$\mathsf{J}\overline{\mathsf{W}} = \overline{\mathsf{W}}\mathsf{J} = -\overline{w}\mathsf{I_d}$$

and

$$\widehat{\overline{\mathsf{W}}}^2 = \mathsf{R}\overline{\mathsf{W}}^2\,^{\mathrm{T}}\mathsf{R} - (2\overline{w} - \omega)\omega\mathsf{I_d}.$$

Let us see now how each term of (19) containing $\overline{\mathsf{W}}$ varies.

From the last equation, we deduce the objectivity of $(\overline{\mathsf{W}}^2)^d$:

$$\widehat{\big(\overline{\mathsf{W}}^2\big)^d} = \mathsf{R}\big(\overline{\mathsf{W}}^2\big)^{d}\,^{\mathrm{T}}\mathsf{R}.$$

For the forth term of (19), we already have (29). And for the last term,

$$\widehat{\bar{\mathsf{S}}}^2\widehat{\overline{\mathsf{W}}} - \widehat{\bar{\mathsf{S}}}\widehat{\overline{\mathsf{W}}}^2 = \mathsf{R}\big(\bar{\mathsf{S}}^2\overline{\mathsf{W}} - \bar{\mathsf{S}}\overline{\mathsf{W}}^2\big)^{\mathrm{T}}\mathsf{R} - \omega\mathsf{R}\bar{\mathsf{S}}^2\,^{\mathrm{T}}\mathsf{R}\mathsf{J} - (2\overline{w} - \omega)\omega\mathsf{R}\bar{\mathsf{S}}^{\mathrm{T}}\mathsf{R}.$$

Putting these results together, we have:

$$\widehat{\mathsf{T}}_s^d = \mathsf{R}\mathsf{T}_s^{d\,\mathrm{T}}\mathsf{R} - \omega\bar{\delta}^2\mathsf{R}\left[2C_4\bar{\mathsf{S}}\mathsf{J} - C_5\frac{1}{|\bar{\mathsf{S}}|}\big(\bar{\mathsf{S}}^2\mathsf{J} - (2\overline{w} - \omega)\omega\bar{\mathsf{S}}\big)\right]^{\mathrm{T}}\mathsf{R}.$$

We conclude that Lund–Novikov model is not invariant. This ends the analysis.

Table 1 summarizes the results of the above analysis. “Y” means that the model is invariant under all the symmetries of the category, “***N***” the opposite and “$\boldsymbol{Y}^*$” that the model is invariant if the conditions (26) on the test filter is verified. It can be seen on it that only two models among the nine, the dynamic and the similarity models, are invariant under all known symmetry group of the NS equations. It is not surprising that the similarity model is invariant since it has a similar form to the real subgrid stress tensor, if it is assumed that the test filter do not destroy any symmetry property. As for the dynamic model, in fact, the dynamical procedure restores the scale invariance of any model [8].

The breaking of symmetries has negative consequences on the ability of the model in correctly representing the flow. For example, when the scaling transformations are broken, important self-similar solutions which like wall laws and vortex solutions cannot be captured. To illustrate this, we will show that the Smagorinsky model which is not invariant under the first scaling transformations (7) cannot reproduce self-similar solutions of NS equations related to these transformations.

The invariants with respect to the scaling transformation (7) are

$$\boldsymbol{\eta} = \frac{\boldsymbol{x}}{t^{1/2}}, \qquad \boldsymbol{u}^* = \frac{\boldsymbol{u}}{t^{-1/2}}, \qquad p^* = \frac{p}{t^{-1}}. \tag{30}$$

These quantities are invariant in the sense that when the transformation (7) is applied to them, they are not altered:

$$\hat{\boldsymbol{\eta}} = \boldsymbol{\eta}, \qquad \widehat{\boldsymbol{u}^*} = \boldsymbol{u}^*, \qquad \widehat{p^*} = p^*.$$

Table 1
Results of the model analysis. Y = invariant, ***N*** = not invariant, ***Y****** = invariant if (26) is verified

	Translations	Rotation, reflection	Scaling transformations	2D material indifference
Smagorinsky	Y	Y	***N***	Y
Dynamic	Y	Y	Y	***Y****
Structure function	Y	Y	***N***	***N***
Gradient	Y	Y	***N***	***N***
Taylor	Y	Y	***N***	***N***
Rational	Y	Y	***N***	***N***
Similarity	Y	Y	Y	***Y****
Lund	Y	Y	***N***	***N***
Kosovic	Y	Y	***N***	***N***

Taken as the new variables, these quantities permit to reduce the NS equations. Indeed, when we introduce the following expressions

$$\boldsymbol{u} = t^{-1/2}\boldsymbol{u}^*(\boldsymbol{\eta}), \qquad \boldsymbol{p} = t^{-1}\boldsymbol{p}^*(\boldsymbol{\eta}) \tag{31}$$

in the NS equations, we obtain a new equation where the number of variables is reduced:

$$-\frac{1}{2}\boldsymbol{u}^* - \boldsymbol{\eta}\nabla_{\boldsymbol{\eta}}\boldsymbol{u}^* + \operatorname{div}_{\boldsymbol{\eta}}(\boldsymbol{u}^* \otimes \boldsymbol{u}^*) + \frac{1}{\rho}\nabla_{\boldsymbol{\eta}} p^* = \nu \operatorname{div}_{\boldsymbol{\eta}} \mathsf{S}^*. \tag{32}$$

In this equation, $\nabla_{\boldsymbol{\eta}}$ and $\operatorname{div}_{\boldsymbol{\eta}}$ are respectively the gradient and the divergence operators according to $\boldsymbol{\eta}$ and $\mathsf{S}^* = (\nabla_{\boldsymbol{\eta}}\boldsymbol{u} + {}^{\mathrm{T}}\nabla_{\boldsymbol{\eta}}\boldsymbol{u})/2$. By applying successive reductions, one finds self-similar solutions of the NS equations. It is such a reduction approach which was used by Ünal to show that the NS equations may have solutions owning the $5/3$ spectra of Kolmogorov [17]. Oberlack, applied the reduction approach to the statistical equations to obtain scaling laws [12]. The reduction with other symmetries leads also to interesting results such as vortex solution of the Navier–Stokes equations [16].

When Smagorinsky model is used, the reduction according to the scaling transformations (7) leads to the following equation

$$-\frac{1}{2}\bar{\boldsymbol{u}}^* - \boldsymbol{\eta}\nabla_{\boldsymbol{\eta}}\bar{\boldsymbol{u}}^* + \operatorname{div}_{\boldsymbol{\eta}}(\bar{\boldsymbol{u}}^* \otimes \bar{\boldsymbol{u}}^*) + \frac{1}{\rho}\nabla_{\boldsymbol{\eta}} \bar{p}^* = \nu \operatorname{div}_{\boldsymbol{\eta}} \bar{\mathsf{S}}^* + t^{-1} \operatorname{div}_{\eta}\big[\big(C_S\bar{\delta}\big)^2\big|\bar{\mathsf{S}}^*\big|\bar{\mathsf{S}}^*\big]. \tag{33}$$

This equation has not been reduced because there is still a dependence on the old variable t. We conclude that Smagorinsky model is not able to capture the self-similar solutions of the NS equations of the form (31), such as the self-similar solutions found by Ünal. This behaviour of Smagorinsky model is also transfered to all statistics equations. Consequently, this model cannot capture the scaling laws of the NS equations proposed by Oberlack. In particular, since the wall laws are lost, the use of wall functions may be necessary in order for the model to represent correctly the flow near the wall.

More generally, most of subgrid models (like all of those presented here) are altered in the following way by the scaling transformation (7):

$$\widehat{\mathsf{T}}_s = \mathrm{e}^{na}\mathsf{T}_s, \tag{34}$$

where n is a real constant. For these models, there exists a tensor T^* which is invariant under the first scaling transformation such that

$$\mathsf{T}_s = t^{-n/2}\mathsf{T}^*. \tag{35}$$

With this relation, one obtain the following equation

$$-\frac{1}{2}\bar{\boldsymbol{u}}^* - \boldsymbol{\eta}\nabla_{\boldsymbol{\eta}}\bar{\boldsymbol{u}}^* + \operatorname{div}_{\boldsymbol{\eta}}(\bar{\boldsymbol{u}}^* \otimes \bar{\boldsymbol{u}}^*) + \frac{1}{\rho}\nabla_{\boldsymbol{\eta}} \bar{p}^* = \nu \operatorname{div}_{\boldsymbol{\eta}} \bar{\mathsf{S}}^* + t^{n/2+1} \operatorname{div}_{\eta} T^*. \tag{36}$$

Consequently, the model is able to capture the self-similar solutions of the NS equations only if $n = -2$. This is also the invariance condition under the first scaling transformation (condition (24) with $b = 0$). As conclusion, only invariant models can capture the self-similar solutions of the NS equations relative to the first scaling transformation.

As told above, the dynamic and the similarity models are invariant under all the symmetries of the NS equations. However, these both models have the inconvenience that they necessitate the use of a test filter. Rather restrictive conditions such as (26) are then needed by these models to preserve the symmetry group of NS equations. In addition, the dynamic model does not satisfy the second law of thermodynamics since it can induce a negative dissipation. Indeed, $\nu + \nu_s$ can take a negative value. To avoid this, an *a posteriori* forcing is generally introduced (see, for example, [1]). In practice, it means that when $\nu + \nu_s$ is negative, ν_s is given a value slightly larger than $-\nu$:

$$\nu_s = -\nu(1-\varepsilon),$$

where ε is a positive real number, small against 1.

Face to this lack of invariance of the existing subgrid-scale models and due to the non-conformity with thermodynamical principles, we propose in Section 5 a way of deriving models which respect the symmetry group of the NS equations and in Section 6 models which verify in addition thermodynamics requirements.

5. Invariant subgrid models

Assume that $\bar{\mathsf{S}} \neq 0$. Let T_s be an analytic function of $\bar{\mathsf{S}}$ and eventually other scalar quantities which are invariant under the symmetry group of the NS equations and on which we do not write the dependence for the moment. Hence, we have:

$$\mathsf{T}_s = \mathfrak{F}(\bar{\mathsf{S}}). \tag{37}$$

By this way, the invariance under transformations (3), (4), (6) and the reflection (9) is guaranteed. From this, the Cayley–Hamilton theorem and the invariance under the rotation (5) lead to:

$$\mathsf{T}_s^d = A(\chi,\zeta)\bar{\mathsf{S}} + B(\chi,\zeta)\,\mathrm{Adj}^d\,\bar{\mathsf{S}}, \tag{38}$$

where $\chi = \operatorname{tr}\bar{\mathsf{S}}^2$ and $\zeta = \det\bar{\mathsf{S}}$ are the invariants of $\bar{\mathsf{S}}$, Adj stands for the operator defined by

$$(\mathrm{Adj}\,\bar{\mathsf{S}})\bar{\mathsf{S}} = (\det\bar{\mathsf{S}})\mathsf{I_d}, \tag{39}$$

($\mathrm{Adj}\,\bar{\mathsf{S}}$ is simply the comatrix of $\bar{\mathsf{S}}$) and A and B are arbitrary scalar functions. Contrarily to the Lund–Novikov model, these coefficient functions will not be taken constant.

Remark that $\mathrm{Adj}^d\,\bar{\mathsf{S}}$ can be replaced by $(\bar{\mathsf{S}}^2)^d$ (they are equal when $\operatorname{tr}\bar{\mathsf{S}} = 0$). However, the relation (39) will be useful in Section 6 and $\mathrm{Adj}\,\bar{\mathsf{S}}$ is computed faster than $\bar{\mathsf{S}}^2$.

Next, T_s is invariant under the first scaling transformation (7) if

$$\widehat{\mathsf{T}}_s = \mathrm{e}^{-2a}\mathsf{T}_s. \tag{40}$$

A sufficient condition to verify this relation is that A and B are such that:

$$A(\mathrm{e}^{-4a}\chi, \mathrm{e}^{-6a}\zeta) = A(\chi,\zeta), \qquad B(\mathrm{e}^{-4a}\chi, \mathrm{e}^{-6a}\zeta) = \mathrm{e}^{2a}B(\chi,\zeta),$$

since

$$\widehat{\bar{\mathsf{S}}} = \mathrm{e}^{-2a}\bar{\mathsf{S}} \quad \text{and} \quad \widehat{\mathrm{Adj}\,\bar{\mathsf{S}}} = \mathrm{e}^{-4a}\,\mathrm{Adj}\,\bar{\mathsf{S}}.$$

Differentiating according to a and taking $a = 0$, it follows:

$$-4\chi\frac{\partial A}{\partial\chi} - 6\zeta\frac{\partial A}{\partial\zeta} = 0, \qquad -4\chi\frac{\partial B}{\partial\chi} - 6\zeta\frac{\partial B}{\partial\zeta} = 2B.$$

To satisfy these equalities, one can take

$$A(\chi,\zeta) = A_1\left(\frac{\zeta}{\chi^{3/2}}\right), \qquad B(\chi,\zeta) = \frac{1}{\sqrt{\chi}}B_1\left(\frac{\zeta}{\chi^{3/2}}\right),$$

where A_1 and B_1 are arbitrary scalar functions. Finally, if $v = \zeta/\chi^{3/2}$ then

$$\mathsf{T}_s^d = A_1(v)\bar{\mathsf{S}} + \frac{1}{\sqrt{\chi}}B_1(v)\,\mathrm{Adj}^d\,\bar{\mathsf{S}}. \tag{41}$$

Relation (41) defines a model which is invariant under the second scaling transformation. A_1 and B_1 have the dimension of a viscosity and v is dimensionless. In order to have the proper dimensions, let us introduce the turbulent subgrid-scale energy q and the dissipation rate ε, defined by:

$$q = \frac{1}{2}\overline{\boldsymbol{u}'^2} \quad \text{and} \quad \varepsilon = 2\nu\,\mathrm{tr}\big[\overline{\mathsf{S}'^2}\big].$$

Since q^2/ε has the dimension of a viscosity, we can choose that:

$$A_1(v) = \frac{q^2}{\varepsilon}A_2(v) \quad \text{and} \quad B_1(v) = \frac{q^2}{\varepsilon}B_2(v),$$

where A_2 and B_2 are dimensionless arbitrary scalar functions.

As mentioned by Oberlack [19], $\boldsymbol{u}'$ transforms in the same way than $\bar{\boldsymbol{u}}$ under the symmetries of the NS equations, except under the generalized Galilean transformation (6) and the 2D material transformation (10) for which, respectively:

$$\widehat{\boldsymbol{u}'} = \boldsymbol{u}',$$
$$\widehat{\boldsymbol{u}'} = \mathsf{R}\boldsymbol{u}'$$

(remark that $\widehat{\bar{\boldsymbol{u}} + \boldsymbol{u}'}$ must be equal to $\hat{u}$). It can then be shown that each symmetry of the NS equations leaves q and ε invariant, except the two scaling transformations. However, the ratio q^2/ε is invariant under the first scaling transformation. Hence, the final model defined by

$$\boxed{\mathsf{T}_s^d = \frac{q^2}{\varepsilon}\left(A_2(v)\bar{\mathsf{S}} + \frac{1}{\sqrt{\chi}}B_2(v)\,\mathrm{Adj}^d\,\bar{\mathsf{S}}\right)} \tag{42}$$

verify (40) and is then invariant.

It remains the second scaling transformation for which the model is invariant if and only if

$$\widehat{\mathsf{T}}_s = \mathrm{e}^{2a}\mathsf{T}.$$

Since $\widehat{\bar{\mathsf{S}}} = \bar{\mathsf{S}}$, $\hat{q} = \mathrm{e}^{2a}$ and $\hat{\varepsilon} = \mathrm{e}^{2a}\varepsilon$, this condition is automatically verified.

To sum up, relation (42) defines a class of subgrid models which are invariant under all symmetry groups of the NS equations. In the next section, this class will be refined to models which are consistent with the second law of thermodynamics.

6. Consequences of the second law of thermodynamics

T_s represents the energy exchange between resolved and subgrid scales. It generates certain dissipation, which, however, may take a negative value (in the case of a backscatter). Nevertheless, in order to respect the second law of thermodynamics, we must ensure that the total dissipation (sum of the molecular and the subgrid dissipations) remains positive.

On molecular scale, the viscous constraint is:

$$\rho\mathsf{T} = \rho\frac{\partial\psi}{\partial\mathsf{S}},$$

where $\psi = \nu\,\mathrm{tr}\,\mathsf{S}^2 = \nu\chi$ is a “potential”. This “potential” form is important because the convexity and positivity of ψ ensures that the molecular dissipation is positive:

$$\Phi = \rho\,\mathrm{tr}(\mathsf{TS}) \geqslant 0.$$

The tensor T_s can be considered as a subgrid constraint generating dissipation

$$\Phi_s = \rho\,\mathrm{tr}\big(\mathsf{T}_s\bar{\mathsf{S}}\big).$$

To remain compatible with the NS equations, we assume that T_s has the same form as T, i.e.:

$$\rho\mathsf{T}_s = \rho\frac{\partial\psi_s}{\partial\bar{\mathsf{S}}}, \tag{43}$$

where ψ_s is a "potential" depending on the invariants χ and ζ of $\bar{\mathsf{S}}$. In this way, the constraint tensor of the filtered equations, $\bar{\mathsf{T}} + \mathsf{T}_s$, has the same "potential" form as the constraint tensor of the NS equations. This hypothesis refines class (41) in the following way.

One deduces from (43) that

$$\mathsf{T}_s^d = 2\frac{\partial \psi_s}{\partial \chi}\bar{\mathsf{S}} + \frac{\partial \psi_s}{\partial \zeta}\,\mathrm{Adj}^d\,\bar{\mathsf{S}}.$$

Comparing this with (41), we obtain:

$$\frac{q^2}{\varepsilon}\frac{1}{2}A_2(v) = \frac{\partial \psi_s}{\partial \chi}, \qquad \frac{q^2}{\varepsilon}\frac{1}{\sqrt{\chi}}B_2(v) = \frac{\partial \psi_s}{\partial \zeta}.$$

Thus, compatibility between the latest two equations requires:

$$\frac{\partial}{\partial \zeta}\left(\frac{1}{2}A_2(v)\right) = \frac{\partial}{\partial \chi}\left(\frac{1}{\sqrt{\chi}}B_2(v)\right).$$

If g is a primitive of B_2, a solution of this equation is

$$A_2(v) = 2g(v) - 3v\dot{g}(v) \quad \text{and} \quad B_2(v) = \dot{g}(v). \tag{44}$$

Then, hypothesis (43) induces the existence of a real-valued dimensionless scalar continued function g such that:

$$\boxed{\mathsf{T}_s^d = \frac{q^2}{\varepsilon}\left[\big(2g(v) - 3v\dot{g}(v)\big)\bar{\mathsf{S}} + \frac{1}{\sqrt{\chi}}\dot{g}(v)\,\mathrm{Adj}^d\,\bar{\mathsf{S}}\right].} \tag{45}$$

Now, let Φ_T be the total dissipation. We have:

$$\Phi_T = \rho\,\mathrm{tr}\big[\big(\bar{\mathsf{T}} + \mathsf{T}_s\big)\bar{\mathsf{S}}\big].$$

Using (41) and (44), it can be shown that $\Phi_T \geqslant 0$ if and only if

$$\boxed{\nu + \frac{q^2}{\varepsilon}g(v) \geqslant 0.} \tag{46}$$

Consequently, if g satisfies the condition (46) then the total dissipation is positive.

In summary, a model belonging to the class (45) with a real continuous dimensionless function g verifying (46) is a model respecting the symmetry group of the NS equations and is conform with the second law of thermodynamics.

In the next subsection, it will be proved that v belongs to a bounded interval. This property may be important for the condition (46) because it enlarges the choice of the function g.

6.1. Bounds of v

Because v is a function of the invariants of $\bar{\mathsf{S}}$, it is independent of the basis. We can then choose a basis in which $\bar{\mathsf{S}}$ is diagonal

$$\bar{\mathsf{S}} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

where $\lambda, \lambda_2, \lambda_3 \in \mathbb{R}$. As $\bar{\mathsf{S}}$ was assumed to be non-zero, it can be supposed that $\lambda \neq 0$. Let $c = \lambda_2/\lambda$. Since $\mathrm{tr}\,\bar{\mathsf{S}} = 0$, we have:

$$\bar{\mathsf{S}} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & c\lambda & 0 \\ 0 & 0 & -(1+c)\lambda \end{pmatrix}.$$

Thereafter,

$$v = \frac{\det \bar{\mathsf{S}}}{(\mathrm{tr}\,\bar{\mathsf{S}}^2)^{3/2}} = \frac{\lambda^3 c(1+c)}{[\lambda^2 + c^2\lambda^2 + (1+2c+c^2)\lambda^2]^{3/2}} = \frac{c(1+c)}{2^{3/2}(1+c+c^2)^{3/2}}.$$

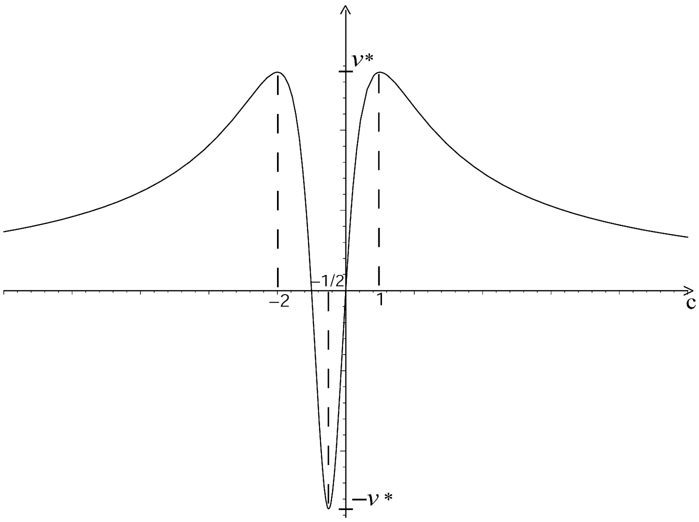

Fig. 1. Aspect of the function h.

The behaviour of the function

$$h : c \longmapsto \frac{c(1+c)}{2^{3/2}(1+c+c^2)^{3/2}}$$

is represented on Fig. 1. A study of its derivative permits to calculate the extrema and to deduce that

$$|v| \leqslant v^* = \frac{1}{3\sqrt{6}}.$$

It proves that if the function g is such that condition (46) is verified for all $v \in [-v^*, v^*]$, then the model defined by (45) respects the second law of thermodynamics.

In the next subsection, we will show that the model we proposed is stable, i.e. have a finite energy. In fact, we will prove a more general result: any model which is symmetric (${}^{\mathrm{T}}\mathsf{T}_s = \mathsf{T}_s$) and which verifies the second law of thermodynamics is stable. Note that, since $\bar{\mathsf{S}}$ is symmetric, it is so for $\operatorname{Adj}\bar{\mathsf{S}}$ and for any model belonging to the class (45).

6.2. The second law of thermodynamics and model stability

Assume that the boundary condition is of a Dirichlet-type. After an appropriate change of variables, the filtered Navier–Stokes equations can be written in the following way:

$$\begin{cases} \dfrac{\partial \bar{\boldsymbol{u}}}{\partial t} + \operatorname{div}(\bar{\boldsymbol{u}} \otimes \bar{\boldsymbol{u}}) + \dfrac{1}{\rho}\nabla \bar{p} = \operatorname{div}\big(\bar{\mathsf{T}} + \mathsf{T}_s\big) + \boldsymbol{F}, \\ \operatorname{div} \bar{\boldsymbol{u}} = 0 \end{cases} \tag{47}$$

with the boundary and initial conditions:

$$\begin{cases} \bar{\boldsymbol{u}} = 0 & \text{on } \partial\Omega, \\ \bar{\boldsymbol{u}}(0, \boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x}) & \text{on } \Omega, \end{cases} \tag{48}$$

where Ω is the domain, $\partial\Omega$ its boundary, $\boldsymbol{F} = \boldsymbol{F}(t, \boldsymbol{x})$ a vector arisen from the variable change and $t_f > 0$ the final time of observation.

The following proposition stipulates that if the model is symmetric and verify the second law of thermodynamics then the energy is finite.

Proposition 1. *Let* $(\bar{\boldsymbol{u}}, \bar{p})$ *a regular solution of* (47), (48) *where* T_s *is a symmetric tensor such that*

$$\operatorname{tr}\big[\big(\bar{\mathsf{T}} + \mathsf{T}_s\big)\bar{\mathsf{S}}\big] \geqslant 0.$$

Then,

$$\left\|\bar{\boldsymbol{u}}(t,\boldsymbol{x})\right\|_{L^2(\Omega)} \leqslant \left\|\boldsymbol{u}_0(\boldsymbol{x})\right\|_{L^2(\Omega)} + \int_0^{t_f} \left\|F(s,\boldsymbol{x})\right\|_{L^2(\Omega)} \mathrm{d}s.$$

Proof. Let us note the scalar product of $L^2(\Omega)$ by $(\cdot,\cdot)$.

Testing the momentum equation of (47) by $\bar{\boldsymbol{u}}$ and using the boundary condition, we obtain:

$$\left(\frac{\partial \bar{\boldsymbol{u}}}{\partial t}, \bar{\boldsymbol{u}}\right) + b(\bar{\boldsymbol{u}},\bar{\boldsymbol{u}},\bar{\boldsymbol{u}}) - \frac{1}{\rho}(p, \operatorname{div}\bar{\boldsymbol{u}}) + \left(\overline{\mathsf{T}} + \mathsf{T}_s, \nabla\bar{\boldsymbol{u}}\right) = (\boldsymbol{F}, \bar{\boldsymbol{u}}), \tag{49}$$

where b is the trilinear function defined by:

$$b(\boldsymbol{r},\boldsymbol{v},\boldsymbol{w}) = \left(\operatorname{div}(\boldsymbol{r}\otimes\boldsymbol{v}), \boldsymbol{w}\right).$$

The first term of (49) is equal to:

$$\left(\frac{\partial \bar{\boldsymbol{u}}}{\partial t}, \bar{\boldsymbol{u}}\right) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\bar{\boldsymbol{u}},\bar{\boldsymbol{u}}) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{\boldsymbol{u}}\|^2_{L^2(\Omega)} = \|\bar{\boldsymbol{u}}\|_{L^2(\Omega)}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{\boldsymbol{u}}\|_{L^2(\Omega)}. \tag{50}$$

The second term of (49) vanishes. Indeed, integrations by parts yield:

$$b(\boldsymbol{r},\boldsymbol{v},\boldsymbol{w}) + b(\boldsymbol{r},\boldsymbol{w},\boldsymbol{v}) = -(\operatorname{div}\boldsymbol{r}, \boldsymbol{v}\cdot\boldsymbol{w}) + \int_{\partial\Omega} (\boldsymbol{v}\cdot\boldsymbol{w})(\boldsymbol{u}\cdot\boldsymbol{n})\,\mathrm{d}s,$$

where $\boldsymbol{n}$ is the unit normal vector outward to Ω. Knowing that $\operatorname{div}\bar{\boldsymbol{u}} = 0$ and $\bar{\boldsymbol{u}}_{|\partial\Omega} = 0$, it follows that

$$b(\bar{\boldsymbol{u}},\bar{\boldsymbol{u}},\bar{\boldsymbol{u}}) = 0. \tag{51}$$

Since $\operatorname{div}\bar{\boldsymbol{u}} = 0$, the third term of (49) also vanishes:

$$\frac{1}{\rho}(p, \operatorname{div}\bar{\boldsymbol{u}}) = 0. \tag{52}$$

Next, because T and T_s are symmetric,

$$(\mathsf{T} + \mathsf{T}_s, \nabla\bar{\boldsymbol{u}}) = \left(\mathsf{T} + \mathsf{T}_s, \bar{\mathsf{S}}\right). \tag{53}$$

Lastly, the right-hand side of (49) verifies:

$$(\boldsymbol{F}, \bar{\boldsymbol{u}}) \leqslant \|\boldsymbol{F}\|_{L^2(\Omega)}\|\bar{\boldsymbol{u}}\|_{L^2(\Omega)}. \tag{54}$$

So, introducing (50)–(53) and (54) in (49), one obtains:

$$\|\bar{\boldsymbol{u}}\|_{L^2(\Omega)}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{\boldsymbol{u}}\|_{L^2(\Omega)} + \left(\overline{\mathsf{T}} + \mathsf{T}_s, \bar{\mathsf{S}}\right) \leqslant \|\boldsymbol{F}\|_{L^2(\Omega)}\|\bar{\boldsymbol{u}}\|_{L^2(\Omega)}.$$

Since

$$\operatorname{tr}\left[\left(\overline{\mathsf{T}} + \mathsf{T}_s\right)\bar{\mathsf{S}}\right] \geqslant 0,$$

this equation gives:

$$\|\bar{\boldsymbol{u}}\|_{L^2(\Omega)}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{\boldsymbol{u}}\|_{L^2(\Omega)} \leqslant \|\boldsymbol{F}\|_{L^2(\Omega)}\|\bar{\boldsymbol{u}}\|_{L^2(\Omega)}.$$

Finally, simplifying by $\|\bar{\boldsymbol{u}}\|_{L^2(\Omega)}$ and integrating on $[0,t]$, one has:

$$\left\|\bar{\boldsymbol{u}}(t,\boldsymbol{x})\right\|_{L^2(\Omega)} \leqslant \left\|\boldsymbol{u}_0(\boldsymbol{x})\right\|_{L^2(\Omega)} + \int_0^{t} \left\|\boldsymbol{F}(s,\boldsymbol{x})\right\|_{L^2(\Omega)} \mathrm{d}s \leqslant \left\|\boldsymbol{u}_0(\boldsymbol{x})\right\|_{L^2(\Omega)} + \int_0^{t_f} \left\|\boldsymbol{F}(s,\boldsymbol{x})\right\|_{L^2(\Omega)} \mathrm{d}s.$$

This proves that the energy is finite. □

We will end up with some remarks on the model and an example.

6.3. Remarks and example

For the derivation of the final model defined by (45), it was assumed that $\bar{\mathsf{S}} \neq 0$. It is easy to show that:

$$\|\mathsf{T}_s\| \longrightarrow 0 \quad \text{when } \|\bar{\mathsf{S}}\| \longrightarrow 0$$

since v is bounded. This is an important property in the outer part of a boundary layer flow where $\bar{\boldsymbol{u}}$ is constant, $\|\bar{\mathsf{S}}\|$ is naturally zero.

The next remark is that from the turbulence theory, it is known that

$$\frac{q^2}{\varepsilon} = \mathrm{O}(\ell), \tag{55}$$

where ℓ is the square of $\bar{\delta}$. Hence,

$$\mathsf{T}_s \longrightarrow 0 \quad \text{when } \bar{\delta} \longrightarrow 0.$$

More precisely, $\mathsf{T}_s = \mathrm{O}(\bar{\delta}^2)$. This insures that the solution of the filtered equations (11), with the model (45), tends to the solution of the initial NS equations (1) when the filter width tends to zero.

For the effective calculation, q and ε can be evaluated from turbulence theory or by using a test filter and approximating the subgrid scales $\boldsymbol{u}'$ for example by the smallest resolved scales:

$$\boldsymbol{u}' \simeq \bar{\boldsymbol{u}} - \tilde{\bar{\boldsymbol{u}}}.$$

We will end up with an example of a model of the class (45). For this, we choose a very simple linear form for g:

$$g(v) = Cv, \tag{56}$$

where C is the model constant. The model has then the following expression:

$$\boxed{\mathsf{T}_s^d = C\frac{q^2}{\varepsilon}\left(-\frac{\det\bar{\mathsf{S}}}{\|\bar{\mathsf{S}}\|^3}\bar{\mathsf{S}} + \frac{1}{\|\bar{\mathsf{S}}\|}\,\mathrm{Adj}^d\,\bar{\mathsf{S}}\right).} \tag{57}$$

C can be evaluated in various ways, for example by a spectral analysis or a dynamic procedure. One other technique is presented here.

Let $\langle\varepsilon_t\rangle$ be the energy transfer where the operator $\langle\cdot\rangle$ stands for an ensemble average. One has:

$$\langle\varepsilon_t\rangle = \langle\mathrm{tr}[\mathsf{T}_s\bar{\mathsf{S}}]\rangle = \left\langle 2\frac{q^2}{\varepsilon} g\,\mathrm{tr}\,\bar{\mathsf{S}}^2\right\rangle \simeq \frac{\langle q\rangle^2}{\langle\varepsilon\rangle}\langle g\rangle\langle 2\,\mathrm{tr}\,\bar{\mathsf{S}}^2\rangle. \tag{58}$$

Next, reasoning in the framework of Kolmogorov's hypothesis, we assume that

$$\langle q\rangle = \frac{3}{2}K_0\langle\varepsilon\rangle^{2/3}k_c^{-2/3}, \tag{59}$$

where $K_0 \simeq 1.4$ and k_c is the cut-off length. Using the local equilibrium hypothesis and some approximations, it follows that

$$C \propto \frac{3C_0}{c}, \tag{60}$$

where $C_0 = (3K_0/2)^{-3}$ and

$$c = \frac{\langle\mathrm{tr}\,\bar{\mathsf{S}}^3\rangle}{\langle\mathrm{tr}\,\bar{\mathsf{S}}^2\rangle^{3/2}}.$$

Finally, from approximations of c in [30] and [2] when $k_c \to 0$, one deduces an order of magnitude of C:

$$C \propto 0.648.$$

For a given viscosity ν, the function g, defined by (56), verifies condition (46) when the discretization is fine enough, because of (55). The model (57) is then conform with the second law of thermodynamics.

7. Conclusion

We have shown that many existing models do not respect the symmetries of the NS equations. This is prejudicial for these models because they are not able to capture the information contained in the equations, such as conservation laws, ... Only few models are consistent with the symmetry group of the NS equations (under the condition however that the eventual test filter has good properties). Among them, some are not compatible with thermodynamic principles. To surpass these incompatibilities, we have proposed a new class of models which respect the fundamental properties of Navier–Stokes equations in the point of view of both symmetry theory and thermodynamics. Finally, we have proved that the respect of the second law of thermodynamics leads to the stability of the model equation.

Deeper studies could be carried out for the choice of the function g in (45). This could be done using a spectral or a correlation approach. Razafindralandy, Hamdouni and Béghein carried out a numerical test [32,33] on an invariant and thermodynamics compatible model (with a different form from (45)). They obtained better results than those provided by Smagorinsky and dynamic models in the configuration of an air flow within a ventilated room.

In a forthcoming paper, the analysis and the derivation of heat convection models will be presented.

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